A General Method to Simulate Noise in Oscillators Based on Frequency Domain Techniques

Werner Anzill and Peter Russer

Abstract—**A perturbation theory for simulating the noise behavior in free running microwave oscillators based on a piecewise harmonic balance technique is outlined. The single-sideband phase noise of an oscillator is derived from the system equations describing the deterministic and stochastic behavior. The method is neither limited to a certain circuit topology nor to certain types of noise sources. The theory is applied to a planar integrated microwave oscillator at 14 GHz to demonstrate the applicability of the theory. Simulated and measured single-sideband phase noise agree within the accuracy of measurement.**

I. Introduction

THE SIMULATION of the spectral behavior of microwave and millimeter-wave circuits is of fundamental importance due to the technology of monolithic integration and the limitations of tuning the circuit after its integration. Besides the signal properties the noise behavior is essential for the design of microwave oscillators. While the determination of the steady state of oscillators in the time and frequency domain is state of the art [1]–[5] and already implemented in modern computer-aided design tools, this is not so for the simulation of the noise behavior. For signal and noise analysis of linear circuits the correlation method is used [6]–[10]. These methods are not suitable to analyze oscillator circuits, since the nonlinearities affect the output characteristics of an oscillator.

A traditional technique to describe noise in oscillating systems [11], [12] is based on an approximation of a slowly varying envelope of the oscillator signal. Closed formulas for the steady state and for the noise spectra of oscillators can be found and a good qualitative and sometimes quantitative understanding of the noise behavior is achieved. But the applicability of the theory is restricted to simplified oscillator models, e.g., Van der Pol oscillators. A model of a linear feedback oscillator is used in [13] and a formula for the phase noise is derived with the noise figure of the transistor, the signal power and the loaded quality factor of the resonator as the characteristics. In both methods noise spectra are derived from a linearized, i.e., small-signal analysis, despite the fact that a stable oscillation is only possible in a nonlinear system. Therefore the conversion of a baseband noise to the harmonics and the modulation of the noise sources by the large-signal steady state are determined from a small-signal analysis.

A method to calculate the general correlation spectrum of oscillators in the time domain with white and $f^{-\alpha}$-noise sources has already been published [14]. The oscillator circuit is described by a lumped circuit model containing the inherent noise sources of the oscillator. This method is based on the solution of the Langevin equations, which describe the stochastic behavior. There are no restrictions on the complexity and the nonlinearities contained in the model. The modulation of the noise sources by the unperturbed oscillation are taken into account.

For microwave oscillators with distributed elements, mainly frequency domain methods are used due to the difficulty of describing distributed elements in the time domain. The calculation of the oscillator's noise behavior in the frequency domain is based on conversion matrices [15], [16], as used for mixers [17]–[20]. The noise signals are described as a superposition of several sinusoidal time functions with different frequencies. The shortcoming of [15] is that only the fundamental frequency of the signal is taken into account and in [16] the nonlinear noise current sources are connected across the nonlinearities that are only one-ports. Both methods require an inversion of the conversion matrix that is ill conditioned in the vicinity of the steady state and the frequency of oscillation. This is an inherent problem of oscillators due to the lack of phase reference. Therefore the phase noise computations of complex oscillators, which have to be done numerically, turn out to be very sensitive to numerical errors.

We propose an approach based on a piecewise harmonic balance technique to calculate the single-sideband phase noise of oscillators that is neither limited to a certain kind of topology of the circuit nor to special characteristics of the noise sources. In particular, the technically important computation of the oscillator noise spectra near the frequency of oscillation is an inherently ill conditioned problem when performed by direct inversion of the corresponding linearized equations [21]. In oscillators the noise sources are small compared with the signals, if the oscillator is not operated in the neighborhood of a bifurcation point. Therefore, the system equations are linearized around the steady state. Due to the lack of phase reference in oscillators the resulting Jacobian is singular at

Manuscript received March 25, 1993; revised June 16, 1993. This work was supported by the Deutsche Forschungsgemeinschaft.
W. Anzill is with the Lehstuhl für Hochfrequenztechnik, Technische Universität, 8000 München 2, Germany.
P. Russer is with the Lehrstuhl für Hochfrequenztechnik, Technische Universität, Müchen and with the Ferdinand-Braun-Institut für Höchstfrequenztechnik, O-1199 Berlin, Germany.
IEEE Log Number 9213000.

0018-9480/93$03.00 © 1993 IEEE

the steady state [22] and ill conditioned for a small frequency deviation from the carrier frequency, where we are interested in the phase noise. We overcome this problem by using an eigenvalue decomposition of the Jacobian, where the small eigenvalue responsible for the bad condition of the matrix is taken into account. The correlation spectrum of the state variable fluctuations is derived, where the phase noise, the amplitude noise, and the amplitude phase correlation spectrum are included. The phase noise is generated by a random phase shift of the unperturbed steady-state signal. Since in oscillators the phase noise is the dominant noise phenomenon we consider only the phase noise correlation spectrum. A simple equation for the simulation of the single-sideband phase noise $L(f_m)$ can be derived that allows to compute $L(f_m)$ in a numerically stable way [23], [24].

In Section II of this paper we formulate the system equations describing the steady state. The method of the noise analysis is outlined in Section III and an expression for the single-sideband phase noise is derived. In Section IV we apply the theory to a planar integrated free running microwave oscillator at 14 GHz.

II. Steady State

To analyze the noise behavior of oscillators we first compute the steady state without the noise sources. As usual for the piecewise harmonic balance method the circuit is divided into a nonlinear and a linear subcircuit. The nonlinear subcircuit is described by the admittance or impedance matrix. The n components of the vector $\mathbf{X}$ are the n state variables that uniquely determine the state of the nonlinear circuit connected with the admittance or impedance of the linear circuit.

$$\mathbf{X} = (\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_n)^T. \tag{1}$$

Each state variable is considered to be periodic at the steady state. Hence $x_i(t)$ is expressed by a Fourier expansion with the coefficients $X_{i,l}$. The frequency range considered is limited to k harmonics,

$$\mathbf{X}_i = (X_{i,-k} \quad X_{i,-k+1} \quad \cdots \quad X_{i,0} \quad \cdots \quad X_{i,+k})^T \tag{2}$$

where $\mathbf{X}_i \in C^{2k+1}$.

Applying Kirchhoff's current and voltage law we obtain a set of n nonlinear equations $\mathbf{F}(\mathbf{X}, \omega)$ for the state variables $\mathbf{X}$ and the frequency of oscillation ω. The steady state is defined by the nonlinear system equation

$$\mathbf{F}(\mathbf{X}^0, \omega_0) \equiv \mathbf{0}. \tag{3}$$

This nonlinear system of equations (3) has an infinite number of solutions $\mathbf{X}^0$, because the phase of oscillation is arbitrary for free running oscillators. To obtain a unique solution the phase has to be fixed by choosing a real or imaginary part of a Fourier coefficient to be zero.

III. Noise Analysis

A. Fluctuations of the State Variables

Taking the noise sources into account, we obtain a nonlinear system of equations including noise.

$$\mathbf{F}(\mathbf{X}_T, \omega, \mathbf{N}_T) = \mathbf{0}. \tag{4}$$

The noise vector $\mathbf{N}_T$ consists of r noise sources at k harmonics, so that $\mathbf{N}_T \in C^{r(2k+1)}$. This system of equations is described in detail for a nodal harmonic balance system in [21], [24]. We use an ansatz where all Fourier coefficients and the frequency of oscillation are noisy and therefore all possible noise processes including the upconversion of $1/f^\alpha$ noise sources and the AM to PM conversion can be taken into account. The index T denotes the time windowed signals as amplitude spectra of random signals may only be defined for time limited probes of the signals [10]. Calculation of the correlation spectra $T \to \infty$ has to be performed again after the ensemble averaging.

In electrical oscillators noise signals are very small compared with the state variables. Therefore it is sufficient to take the noise sources into account up to first order,

$$\mathbf{F}(\mathbf{X}_T, \omega) + \mathbf{G}(\mathbf{X}_T^0, \omega) \cdot \mathbf{N}_T = \mathbf{0} \tag{5}$$

where $\mathbf{G}(\mathbf{X}_T^0, \omega) \in C^{n(2k+1) \times r(2k+1)}$ and

$$\mathbf{G}(\mathbf{X}_T^0, \omega) \equiv \left.\frac{\partial \mathbf{F}(\mathbf{X}_T, \omega)}{\partial \mathbf{N}_T}\right|_{\mathbf{X}_T = \mathbf{X}_T^0, \mathbf{N}_T = \mathbf{0}}. \tag{6}$$

The matrix $\mathbf{G}(\mathbf{X}_T^0, \omega)$ denotes the contribution of the noise sources $\mathbf{N}_T$ to each equation in (5).

Due to the small noise signals the state variables and the frequency of oscillation deviate only by a small amount from the steady state,

$$\mathbf{X}_T(\omega) = \mathbf{X}_T^0(\omega) + \delta\mathbf{X}_T(\omega); \qquad \omega = \omega_0 + \omega_m$$
$$\|\delta\mathbf{X}_T(\omega)\| \ll \|\mathbf{X}_T^0(\omega)\|; \qquad \omega_m \ll \omega_0. \tag{7}$$

Thus the system of nonlinear equations can be linearized around the steady state,

$$\mathbf{J}(\mathbf{X}_T^0, \omega)\delta\mathbf{X}_T + \mathbf{G}(\mathbf{X}_T^0, \omega)\mathbf{N}_T = \mathbf{0} \tag{8}$$

with $\mathbf{J}(\mathbf{X}_T^0, \omega) \in C^{n(2k+1) \times n(2k+1)}$ and

$$\mathbf{J}(\mathbf{X}_T^0, \omega) \equiv \left.\frac{\partial \mathbf{F}(\mathbf{X}_T, \omega)}{\partial \mathbf{X}_T}\right|_{\mathbf{X}_T = \mathbf{X}_T^0}. \tag{9}$$

The matrix $\mathbf{J}(\mathbf{X}_T^0, \omega)$ represents the Jacobian that contains all information about the noise signals mixed with the spectral components of the state variables $\mathbf{X}_T^0$. The Jacobian is singular at the steady-state $\mathbf{X}_T^0$ and ω_0. That means one eigenvalue is zero, which is denoted $\lambda_1 = 0$. Therefore a distortion $\delta\mathbf{X}_T$ exists, that $\mathbf{X}_T^0 + \delta\mathbf{X}_T$ is also a solution of (3). In other words $\mathbf{X}_T^0 + \delta\mathbf{X}_T$ deviates from the steady-state solution of the oscillator only by a small phase shift. These stochastic phase deviations constitute the phase noise.

B. Solution of the System Equations Including Noise

The Jacobian is singular at the steady state and for a small frequency deviation f_m of the carrier frequency the deviations of the matrix elements are small and the condition number of the Jacobian remains high [25]. The condition number of a matrix can be approximated by the ratio of the largest to the smallest eigenvalue. The largest eigenvalue is much larger than the frequency of oscillation f_0, because it is related to the fastest process of the system. The smallest eigenvalue is in the order of the frequency deviation f_m, as we will show later in (27). Therefore the condition number cond of the Jacobian is much larger than the ratio of the carrier frequency to the frequency deviation of interest [23].

$$\text{cond}\,(\mathbf{J}[\mathbf{X}_T^0, 2\pi(f_0 + f_m)]) \gg \frac{f_0}{f_m}. \tag{10}$$

That means the steady state of oscillators has to be determined to a much higher precision than the inverse of the condition number to achieve a relative error smaller than 1 [26]. Considering a 10 GHz oscillator and a frequency deviation of, e.g., $f_m = 10$ kHz the condition number is much larger than 10^6.

To overcome the numerical problems the Jacobian is linearized at the carrier frequency with respect to the frequency

$$\mathbf{J}(\mathbf{X}_T^0, \omega) = \mathbf{J}(\mathbf{X}_T^0, \omega_0) + \omega_m \cdot \mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0) \tag{11}$$

with abbreviation

$$\mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0) \equiv \left.\frac{\partial \mathbf{J}(\mathbf{X}_T^0, \omega)}{\partial \omega}\right|_{\omega = \omega_0}. \tag{12}$$

Then, an eigenvalue decomposition [27] of the Jacobian with left- and right-sided eigenvectors is used. Thus the complete correlation spectra can be calculated in a numerically stable way.

First we want to analyze the unperturbed Jacobian $\mathbf{J}(\mathbf{X}_T^0, \omega_0)$. The left- and right-sided eigenvectors of the Jacobian are denoted with $\mathbf{V}_j$ and $\mathbf{W}_i$ and the eigenvalues with λ_j and λ_i respectively.

$$\mathbf{V}_j^+ \cdot \mathbf{J}(\mathbf{X}_T^0, \omega_0) = \lambda_j^V \cdot \mathbf{V}_j^+; \qquad \mathbf{V}_j \in C^{n(2k+1)} \tag{13}$$

$$\mathbf{J}(\mathbf{X}_T^0, \omega_0) \cdot \mathbf{W}_i = \lambda_i^W \cdot \mathbf{W}_i; \qquad \mathbf{W}_i \in C^{n(2k+1)}. \tag{14}$$

The eigenvalues of the Jacobian are equal for a set of left- and right-sided eigenvectors.

$$\lambda_j^V = \lambda_i^W = \lambda_i \qquad \text{for } i = j. \tag{15}$$

The left- and right-sided eigenvectors satisfy the orthogonality relations [25]:

$$\mathbf{V}_j^+ \cdot \mathbf{W}_i = \delta_{ij} \quad \text{with} \quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{16}$$

These equations mean, e.g., the eigenvector $\mathbf{V}_1$ is orthogonal to all right-sided eigenvectors $\mathbf{W}_i$ with the exception of $\mathbf{W}_1$. The eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$ are denoted with $\mathbf{V}_1$ and $\mathbf{W}_1$. These are the eigenvectors that we will need later on.

The eigenvector $\mathbf{W}_1$ is determined by the steady state [21], [24].

$$\mathbf{W}_1 = j\omega_0 \mathbf{K}\mathbf{X}_T^0 \tag{17}$$

where $\mathbf{K} \in R^{n(2k+1) \times n(2k+1)}$ is a matrix that has only nonvanishing diagonal elements consisting of the number of the harmonics.

$$\mathbf{K} = \begin{bmatrix} -k & & & & & & \\ & -(k-1) & & & & \mathbf{0} & \\ & & \ddots & & & & \\ & & & +k & & & \\ & & & & -k & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & k \end{bmatrix}. \tag{18}$$

The physical meaning of the vectors $\mathbf{V}_1$ and $\mathbf{W}_1$ is illuminated in the time domain; see Fig. 1.

$\mathbf{w}_1(t) \circ\!\!-\!\!\bullet\ \mathbf{W}_1$ is the tangent vector to the steady-state $\mathbf{x}^0(t)$ and $\boldsymbol{v}_1(t) \circ\!\!-\!\!\bullet\ \mathbf{V}_1$ is the normal vector defining a plane $\mathcal{N}$ that is mapped onto itself by the unperturbed flux of the linearized set of differential equations (Poincaré map), see [14].

The vector $\mathbf{V}_1$ is determined by the definition

$$\mathbf{J}^+(\mathbf{X}_T^0, \omega_0) \cdot \mathbf{V}_1 = \mathbf{0} \tag{19}$$

which is a linear homogeneous system of equations and can be solved with a standard LU-decomposition. The length of the vector $\mathbf{V}_1$ has to be normalized to satisfy (16).

$$\|\mathbf{V}_1\|_2 = \|\omega_0 \mathbf{K}\mathbf{X}_T^0\|_2^{-1} \tag{20}$$

The eigenvectors $\mathbf{W}_i$ are a complete base of the state space and due to (16) a multiplication of $\mathbf{V}_1^+$ with a vector within the state space is a projection onto the complementary space of the plane $\mathcal{N}$. That means, the projection operator $\mathbf{W}_1\mathbf{V}_1^+$ applied to a vector, e.g., named $\mathbf{z} = \sum_{i=1}^n a_i \mathbf{W}_i$, results in a vector tangential to the limit cycle with a length of the coefficient a_1. So if this projection operator $\mathbf{W}_1\mathbf{V}_1^+$ is applied to the noise sources in the state space $\mathbf{G}(\mathbf{X}_T^0, \omega)\mathbf{N}_T$ the contributions of the noise sources that cause a phase shift of the unperturbed steady state are separated. This will be shown by an algebraic derivation in the following part of the paper.

For a small frequency deviation of $\omega_m = 2\pi f_m$ the deviations of the elements of the Jacobian are small. Therefore the deviations of the eigenvalues and eigenvectors are small too, because they are continuous functions of the matrix elements [27].

$$\lambda_i' = \lambda_i + \delta\lambda_i; \qquad |\delta\lambda_i| \ll |\lambda_i| \tag{21}$$

$$\mathbf{V}_j'^+ = \mathbf{V}_j^+ + \delta\mathbf{V}_j^+; \qquad \|\delta\mathbf{V}_j\|_2 \ll \|\mathbf{V}_j\|_2 \tag{22}$$

$$\mathbf{W}_i' = \mathbf{W}_i + \delta\mathbf{W}_i; \qquad \|\delta\mathbf{W}_i\|_2 \ll \|\mathbf{W}_i\|_2. \tag{23}$$

The eigenvalues and eigenvectors of the perturbed Jacobian $\mathbf{J}(\mathbf{X}_T^0, \omega)$ are denoted with a prime. It is sufficient to consider the deviations of the eigenvalues and eigenvectors up to the first order in ω_m.

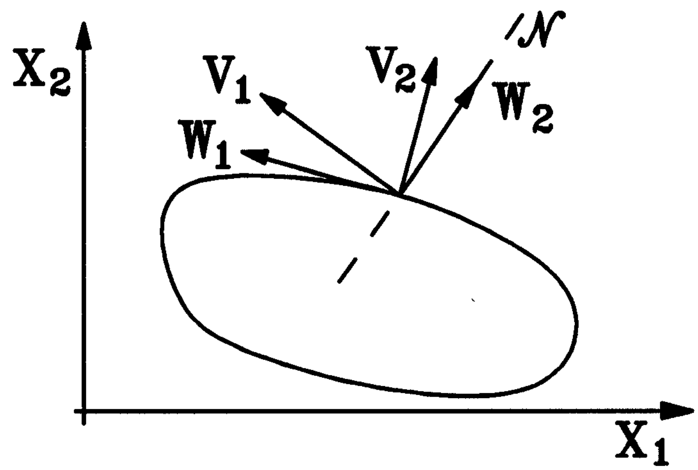


Fig. 1. Two-dimensional phase space with a limit cycle and eigenvectors $\mathbf{v}_i(t)$ and $\mathbf{w}_i(t)$.

$$\delta\lambda_i = \omega_m \mathbf{V}_i^+ \mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0)\mathbf{W}_i \tag{24}$$

$$\delta\mathbf{W}_i = \sum_{l=1, l\neq i}^{n(2k+1)} \frac{\omega_m}{\lambda_i - \lambda_l} \cdot \mathbf{V}_l^+ \mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0)\mathbf{W}_i \cdot \mathbf{W}_l \tag{25}$$

$$\delta\mathbf{V}_j = \sum_{l=1, l\neq j}^{n(2k+1)} \frac{\omega_m}{\lambda_j - \lambda_l} \cdot \mathbf{V}_j^+ \mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0)\mathbf{W}_l \cdot \mathbf{V}_l. \tag{26}$$

Therefore the eigenvalue λ_1' of the perturbed Jacobian $\mathbf{J}(\mathbf{X}_T^0, \omega)$ is given with (17) by

$$\lambda_1' = \delta\lambda_1 = 2\pi f_m \mathbf{V}_1^+ \mathbf{J}_\omega(\mathbf{X}_T^0, \omega_0) j2\pi f_0 \mathbf{K}\mathbf{X}_T^0. \tag{27}$$

The inverse of the Jacobian $\mathbf{J}^{-1}(\mathbf{X}_T^0, \omega)$ is represented by an eigenvalue decomposition with the eigenvalues and left- and right-sided eigenvectors of the Jacobian $\mathbf{J}(\mathbf{X}_T^0, \omega)$.

$$\mathbf{J}^{-1}(\mathbf{X}_T^0, \omega) = \sum_{i=1}^{n} \frac{1}{\lambda_i'} \mathbf{W}_i' \mathbf{V}_i'^+. \tag{28}$$

This inversion will not be calculated due to the bad condition of the Jacobian. We derive this equation to calculate the correlation spectrum of the state variable fluctuations. Later on we take into account the special eigenvalue λ_1' that causes the bad condition of the matrix and the problems of a numerical inversion. The state variable fluctuations are given by

$$\delta\mathbf{X}_T = \sum_{i=1}^{n(2k+1)} \frac{1}{\lambda_i'} \mathbf{W}_i' \mathbf{V}_i'^+ \cdot (-\mathbf{G}(\mathbf{X}^0, \omega)\mathbf{N}_T). \tag{29}$$

C. Correlation Spectrum of the Oscillator Noise

The correlation spectra of the state variables $\mathbf{C}^{\delta X}(f)$ and the noise sources $\mathbf{C}^N(f)$ are given by

$$\mathbf{C}^{\delta X}(f) = \lim_{T\to\infty} \frac{1}{2T} \langle \delta\mathbf{X}_T(f)\delta\mathbf{X}_T^+(f)\rangle \tag{30}$$

$$\mathbf{C}^N(f) = \lim_{T\to\infty} \frac{1}{2T} \langle \mathbf{N}_T(f)\mathbf{N}_T^+(f)\rangle \tag{31}$$

where the brackets denote the ensemble average. The correlation spectra of the state variables are derived using (30), (31) and the equation of the state variable fluctuations (29)

$$\mathbf{C}^{\delta X}(f) = \sum_{i=1}^{n(2k+1)} \sum_{j=1}^{n(2k+1)} 1/(\lambda_i' \lambda_j'^*) \cdot \mathbf{V}_i'^+ \mathbf{C}^{GN}(f)\mathbf{V}_j' \cdot \mathbf{W}_i' \mathbf{W}_j'^+ \tag{32}$$

with abbreviation

$$\mathbf{C}^{GN}(f) = \mathbf{G}(\mathbf{X}_T^0, \omega)\mathbf{C}^N(f)\mathbf{G}^+(\mathbf{X}_T^0, \omega). \tag{33}$$

The approximations of (24)–(26) for the eigenvalues and eigenvectors of the perturbed Jacobian are used to derive the correlation spectra of the state variable fluctuations. The term with the major contribution to the correlation spectrum is the term with $i = j = 1$ due to the small eigenvalue $\lambda_1' = \delta\lambda_1$ given in (27). This term denotes, as already described, the phase noise of oscillators. As the perturbation of the eigenvectors $\delta\mathbf{W}_1$ and $\delta\mathbf{V}_1$ are in the order of ω_m and therefore small compared with the unperturbed eigenvectors, they are negligible.

$$\mathbf{C}^{\delta X}(f) = \frac{\mathbf{V}_1^+ \mathbf{C}^{GN}(f)\mathbf{V}_1 \cdot \mathbf{K}\mathbf{X}^0\mathbf{X}^{0+}\mathbf{K}}{(2\pi f_m)^2 |\mathbf{V}_1^+ \mathbf{J}_\omega(\mathbf{X}^0, \omega_0)\mathbf{K}\mathbf{X}^0|^2}. \tag{34}$$

Due to the special situation of the eigenvalue λ_1' and the eigenvectors $\mathbf{V}_1$ and $\mathbf{W}_1$ the terms with $i = 1$ and $j \neq 1$ or $i \neq 1$ and $j = 1$ in (32) denote the amplitude phase correlation spectra. Finally the terms with $i \neq 1$ and $j \neq 1$ in (32) represent the amplitude noise. These noise contributions are small compared with the phase noise due to the larger eigenvalues and are not taken into account in this paper.

D. Single-Sideband Phase Noise $L(f_m)$

The single-sideband phase noise $L(f_m)$ is the ratio between the noise power in a sideband of bandwidth 1 Hz at a deviation $f_m = f - f_0$ from the carrier frequency and the total signal power P_S. $L(f_m)$ is equal for all state variables and therefore we can choose any state variable x_i to calculate the single-sideband phase noise.

$$L(f_m) = P_{Ni}(f_m)/P_{Si}. \tag{35}$$

In order to obtain the single-sideband phase noise at the fundamental frequency the matrix element corresponding to the ith state variable is chosen that denotes the noise power at the fundamental frequency. We have to select the element $X^0_{i,1}X^{0*}_{i,1} = |X^0_{i,1}|^2$ of the matrix $\mathbf{KX}^0\mathbf{X}^{0+}\mathbf{K}$ and obtain for the noise power $P_{Ni}(f_m)$ in a 1 Hz bandwidth

$$P_{Ni}(f_m) = 2\mathbf{C}^{\delta X}(f_0 + f_m)_{i,1} \cdot R_u = 2 \cdot \frac{\mathbf{V}_1^+ \mathbf{C}^{GN}(f_0 + f_m)\mathbf{V}_1 \cdot |X^0_{i,1}|^2 \cdot R_u}{(2\pi f_m)^2 |\mathbf{V}_1^+ \mathbf{J}_\omega(\mathbf{X}^0, \omega_0)\mathbf{KX}^0|^2}. \tag{36}$$

R_u is a resistance of normalization. The signal power of the fundamental frequency is represented by

$$P_{Si} = 2|X^0_{i,1}|^2 \cdot R_u. \tag{37}$$

With the definition of the single-sideband phase noise in (35) we derive an equation for $L(f_m)$ using the approximations of the noise power (36) and the signal power (37).

$$L(f_m) = \frac{1}{(2\pi f_m)^2} \cdot \frac{\mathbf{V}_1^+ \mathbf{C}^{GN}(f_0 + f_m)\mathbf{V}_1}{|\mathbf{V}_1^+ \mathbf{J}_\omega(\mathbf{X}^0, 2\pi f_0)\mathbf{KX}^0|^2}. \tag{38}$$

$\mathbf{V}_1$ is the solution of a homogeneous linear system of equations, $\mathbf{J}^+(\mathbf{U}^0_T, 2\pi f_0)\mathbf{V}_1 = \mathbf{0}$, which can be obtained very easily with a standard LU-decomposition of the Jacobian. The derivative of the Jacobian with respect to the frequency $\mathbf{J}_\omega(\mathbf{U}^0, 2\pi f_0)$ can be calculated numerically, as we will show in our example. The denominator of the second term is constant for different frequency deviations and needs to be calculated only once. The numerator consists of the correlation spectrum of the noise sources multiplied with the vector $\mathbf{V}_1^+$ from the left side and with $\mathbf{V}_1$ from the right side. As we already described, this multiplication is a projection of all noise sources of the state space onto the tangent vector to the steady state. That means the vector $\mathbf{V}_1$ selects the contributions of the noise sources that are tangential to the steady state and therefore induce the phase noise.

The noise sources, $1/f^\alpha$- and white noise sources, and their modulation are taken into account in the correlation matrix $\mathbf{C}^{GN}$. The correlation spectrum of a $1/f^\alpha$-noise source decreases with $(10 \cdot \alpha)$ dB/frequency decade and therefore $L(f_m)$ decreases at $[20 + (10 \cdot \alpha)]$ dB/decade. The single-sideband phase noise decreases at 20 dB/decade due to the white noise sources, because the correlation spectra of white noise sources are constant with respect to the frequency.

This method results in a numerical stable calculation of the phase noise of free running oscillators, where all effects of the noise sources converted with the harmonic signals are taken into account.

IV. Example

A. Simulation and Measurement of the Single-Sideband Phase Noise of a Planar Integrated Microwave Oscillator at 14 GHz

This new method is applied to a planar [28] integrated microwave oscillator with a GaAs MESFET at 14 GHz to demonstrate the applicability of the theory to technical relevant circuits.

For simulating the phase noise of oscillators a very good model of the transistor and the passive network describing the signal and noise behavior is essential. We therefore developed a signal and noise model of a GaAs MESFET, the NE710. The equivalent circuit of the MESFET (Fig. 2) has been obtained by S-parameter measurements at several bias points.

A modified SPICE model [29], [30] was used to characterize the nonlinearities of the MESFET used. The white noise sources are thermal noise sources of the losses or shot noise sources of the internal diodes of the transistor [31]. The NF-noise power was measured for several bias voltages and the $1/f^\alpha$-nonlinear voltage-controlled current noise source between drain and source was modeled. The measured NF-noise power is depicted in Fig. 3 for a voltage of -0.7 V between gate and source and 3.0 V between drain and source.

The correlation spectrum of the $1/f^\alpha$-noise source is given by

$$\mathbf{C}^f = \frac{c(U_{GS}, U_{DS}) \cdot (10 \text{ kHz})^\alpha}{|f_m|^\alpha}. \tag{39}$$

The function $c(U_{GS}, U_{DS})$ denotes the spectral noise power at a frequency of 10 kHz in dependence of the gate-source and the drain-source voltage. The exponent α was obtained by averaging the slope of the measured baseband noise between 1 and 100 kHz over several bias points.

The linear circuit was designed with microstrip lines for a frequency of oscillation at 14 GHz. The designed circuit is shown in Fig. 4. A photograph of the oscillator is shown in Fig. 5.

The spectrum of the output power measured with the spectrum analyzer HP71000 is shown in Fig. 6 with a maximum power of 12.85 dBm at 14.2 GHz. A 10 dB attenuator was used at the input port of the spectrum analyzer.

The equivalent noise sources at the ports were simulated with the linear network analysis program SANA [32]. Hence the correlation matrices of all noise sources are known. Applying Kirchhoff's voltage and current law in order to obtain the system equations the matrix $\mathbf{G}(\mathbf{U}^0_T, \omega)$ is automatically obtained if the noise sources are taken into account in the equivalent circuit. The vector $\mathbf{V}_1$ is calculated by solving the linear system of equations $\mathbf{J}^+\mathbf{V}_1 = \mathbf{0}$ with a standard LU-decomposition. As the numerical differentiation of the Jacobian with respect to the frequency is not sensitive to the choice of the frequency shift a simple numerical differentiation algorithm can be used. The noise power of the oscillator was measured with the

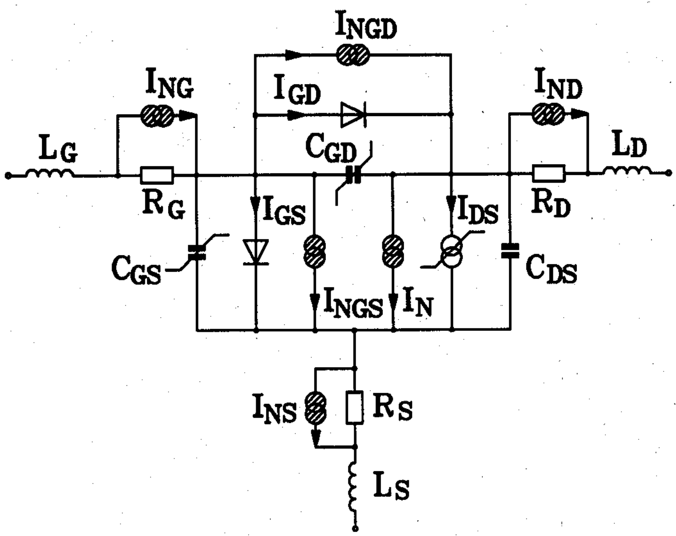


Fig. 2. Equivalent circuit of the GaAs MESFET NE710.

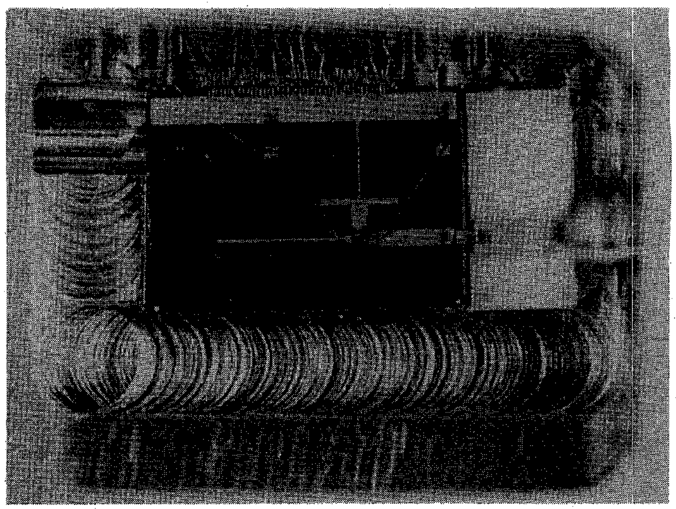


Fig. 5. Photograph of the oscillator.

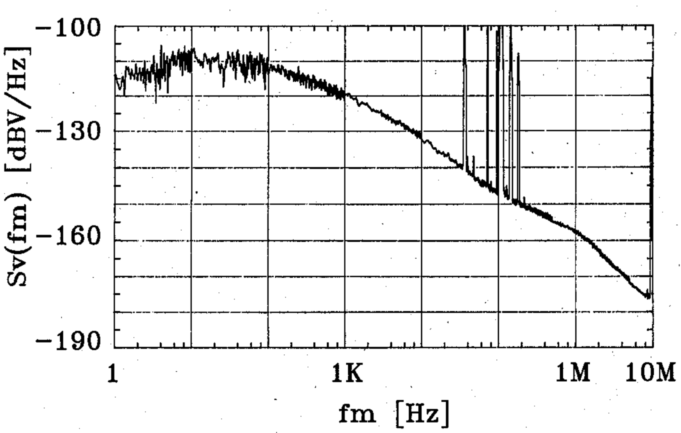


Fig. 3. NF-noise measurement with $U_{GS} = -0.7$ V and $U_{DS} = 3.0$ V.

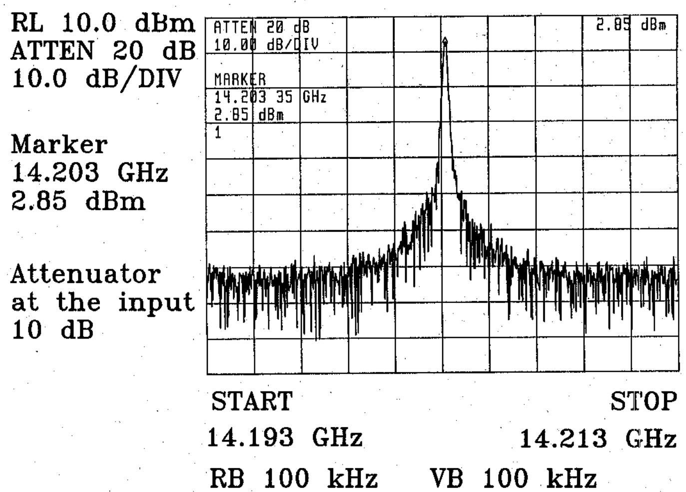


Fig. 6. The output spectrum of the oscillator.

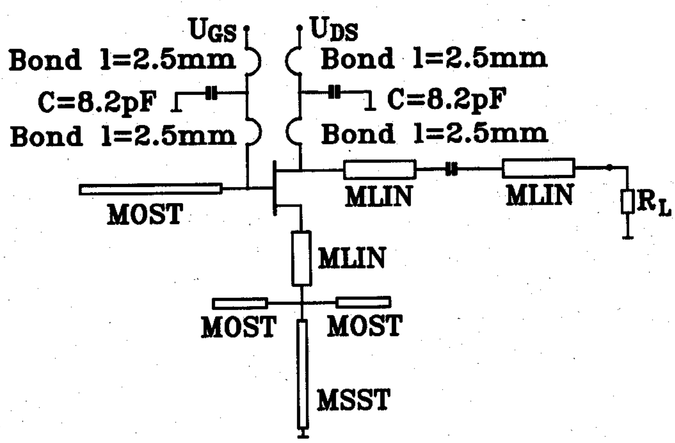


Fig. 4. The oscillator circuit.

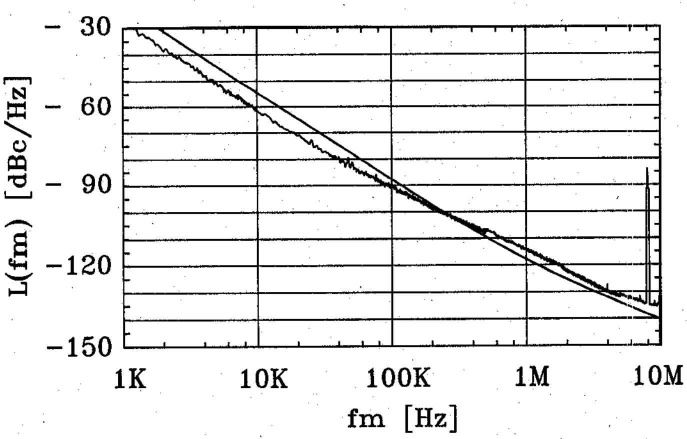


Fig. 7. Measured and simulated single-sideband phase noise $L(f_m)$.

Hewlett Packard HP3048 noise measurement system by using the frequency discriminator method [33]. We obtain a single-sideband phase noise $L(f_m)$ of −90 dBc/Hz at f_m = 100 kHz. The simulated and measured single-sideband phase noise is depicted in Fig. 7, where only one harmonic has been taken into account to simulate $L(f_m)$.

At small frequency deviations the single-sideband phase noise $L(f_m)$ decreases at 33 dB/decade due to the modeled factor $\alpha = 1.3$ of the $1/f^{\alpha}$ noise source. $L(f_m)$ decreases at 20 dB/decade due to the white noise sources for a frequency deviation greater than 1 MHz. The deviation of the simulated and the measured single-sideband phase noise is under 5 dB over the whole measured frequency range from 1 kHz to 10 MHz. Another important feature of our method is the low numerical effort to calculate the noise behavior of oscillators. An HP9000 workstation needs just about 6 s to calculate 50 points of the single-sideband phase noise without any optimization done to minimize the computation time.

V. Conclusion

We demonstrated a numerically stable method to simulate the single-sideband phase noise in free running microwave oscillators based on a piecewise harmonic balance technique. The prerequisites for the calculations are that the steady state of the oscillator without the noise sources describes a limit cycle of the oscillator in the phase space and that the noise sources cause only small deviations from the unperturbed solution. This method takes the conversion of the baseband noise and the conversion of the white noise sources to each harmonic into account. The modulation of the noise sources due to the large-signal steady state are considered.

The procedure described above is also applicable to calculate the amplitude phase correlation spectra and the amplitude noise. These contributions are still taken into account in (32) and can be calculated by choosing the proper terms in the double sum.

The method has been applied to a planar integrated microwave oscillator at 14 GHz. A single sideband phase noise of -90 dBc/Hz at an offset frequency of 100 kHz was obtained by using only microstrip lines at the gate and source as resonators. The difference of the simulated and measured single-sideband phase noise lies within the accuracy of measurements over the whole measured frequency range between 1 kHz and 10 MHz. The method proved to be a fast, reliable, and numerically stable tool for the design of microwave oscillators.

Acknowledgment

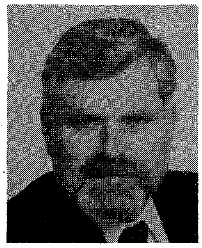
The authors would like to express their appreciation to Dr. F. X. Kärtner for his encouragement and helpful discussions. The authors would also like to thank Dipl.Ing. J. Schaffer and Dr.-Ing. V. Güngerich for designing and measuring the oscillator, and Ms. C. Peterschik, who fabricated the oscillator.

References

[1] T. J. Aprille and T. N. Trick, "A computer algorithm to determine the steady-state response of nonlinear oscillators," *IEEE Trans. Circuit Theory*, vol. 19, pp. 354-360, 1972.
[2] L. O. Chua, *Computer Aided Analysis of Electronic Circuits: Algorithms & Computational Techniques*. Englewood Cliffs, NJ: Prentice Hall, 1975.
[3] M. Schwab, "Determination of the steady state of an oscillator by a combined time-frequency method," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 1391-1402, Aug. 1991.
[4] S. K. Kundert and A. Sangiovanni-Vincentelli. *Steady-State Methods for Simulating Analog and Microwave Circuits*. Boston: Kluwer Academic, 1990.
[5] V. Rizzoli and A. Neri, "State of the art and present trends in nonlinear microwave CAD techniques," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 343-365, Feb. 1988.
[6] A. H. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks*. New York: Wiley, 1985.
[7] P. Russer and H. Hillbrand. "Rauschanalyse von linearen Netzen," *Wis. Ber. AEG-Telefunken*, vol. 49, pp. 127-138, 1976.
[8] H. Hillbrand and P. Russer, "An efficient method for computer aided noise analysis of linear amplifier networks," *IEEE Trans. Circuits Systems*, vol. 23, pp. 235-238, 1976.
[9] S. Heinen, J. Kunisch, and I. Wolff. "Considerations on the analysis of mixers with arbitrary topology including signal and noise properties," in *Proc. IEEE MTT/AP Workshop Integrat. Nonlinear Microwave, Millimeter Circuits (INMMC '90) Dig.*, pp. 63-77, Duisburg, 1990.
[10] P. Russer and S. Müller, "Noise analysis of linear microwave circuits," *Int. J. Numerical Modelling*, vol. 3, pp. 287-316, 1990.
[11] K. Kurokawa, "Some basic characteristics of broadband negative resistance oscillator circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 1937-1955, 1969.
[12] —, "Noise in synchronized oscillators," *IEEE Trans. Microwave Theory Tech.*, vol. 16, pp. 234-240, Apr. 1968.
[13] D. B. Leeson, "A simple model of feedback oscillator noise spectrum," *Proc. IEEE*, pp. 329-330, Feb. 1966.
[14] F. X. Kärtner, "Analysis of white and $f^{-\alpha}$ noise in oscillators," *Int. J. Circuit Theory Applicat.*, vol. 18, pp. 485-519, 1990.
[15] H. J. Siweris and B. Schiek, "Analysis of noise upconversion in microwave FET oscillators," *IEEE Trans. Microwave Theory Tech.*, vol. 33, pp. 233-242, Mar. 1985.
[16] J. M. Paillot, J. C. Nallatamby, M. Hessane, R. Quere, M. Prigent, and J. Rousset, "A general program for steady state, stability, and FM noise analysis of microwave oscillators," in *Proc. IEEE Symp. MTT-S Dig.*, 1990, pp. 1287-1290, Dallas, TX.
[17] D. N. Held and A. R. Kerr, "Conversion loss and noise of microwave and millimeter wave mixers: Part 1—Theory," *IEEE Trans. Microwave Theory Tech.*, vol. 26, pp. 49-55, Feb. 1978.
[18] A. R. Kerr, "Noise and loss in balanced and subharmonically pumped mixers: Part 1—Theory," *IEEE Trans. Microwave Theory Tech.*, vol. 27, pp. 938-943, Dec. 1979.
[19] V. Rizzoli, F. Mastri, and C. Cecchetti, "Computer-aided noise analysis of MESFET and HEMT mixers," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 1401-1410, Sept. 1989.
[20] S. Heinen, J. Kunisch, and I. Wolff, "A unified framework for computer-aided noise analysis of linear and nonlinear microwave circuits," *IEEE Trans Microwave Theory Tech.*, vol. 39, pp. 2170-2175, Dec. 1991.
[21] W. Anzill, F. X. Kärtner, and P. Russer, "Simulation of the phase noise of oscillators in the frequency domain," *Archiv für Elektronik und Übertragungstechnik (AEÜ), Int. J. Electron., Commun.*, 1993.
[22] A. I. Mees, "Limit Cycle Stability," *J. Institut. Maths Applicat.*, vol. 11, pp. 281-295, 1973.
[23] F. X. Kärtner, "Noise in oscillating systems," in *Proc. IEEE MTT/AP Workshop Integrat. Nonlinear Microwave, Millimeter Wave Circuits (INMMC '92)*, Duisburg, 1992, pp. 61-75.
[24] W. Anzill, F. X. Kärtner, and P. Russer, "Simulation of the single-sideband phase noise of oscillators," in *Proc. IEEE MTT/AP Workshop Integrat. Nonlinear Microwave, Millimeter Wave Circuits (INMMC '92)*, Duisburg, 1992, pp. 97-110.
[25] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*. Oxford: Clarendon, 1988.
[26] R. Bulirsch and J. Stoer, *Introduction to Numerical Analysis*. New York: Springer Verlag, 1980.
[27] G. H. Golub and C. T. van Loan, *Matrix Computations*. Baltimore, MD, London: The Johns Hopkins Univ. Press, 1989.
[28] V. Güngerich, R. Schadel, R. Ramisch, and P. Russer, "A process for inserting chips into planar microwave structures on semiconductor substrates," *J. Microelectron. Eng.*, vol. 18, pp. 247-252, 1992.
[29] W. R. Curtice and M. Ettenberg, "A nonlinear GaAs-FET model for use in the design of output circuits for power amplifiers," *IEEE Trans. Microwave Theory Tech.*, vol. 33, pp. 1383-1394, Dec. 1985.
[30] H. Statz, P. Newman, I. W. Smith, R. A. Pucel, and H. A. Haus, "GaAs FET device and circuit simulation in Spice," *IEEE Trans. Electron Devices*, vol. ED-34, pp. 160-169, Feb. 1987.
[31] C. Dragone, "Analysis of thermal and shot noise in pumped resistive diodes," *Bell Syst. Tech. J.*, vol. 47, pp. 1883-1902, Nov. 1968.
[32] *SANA Manual*. RTI GmbH, Munich, 1989.
[33] *HP Product Note 11729C-2: Phase Noise Characterization of Microwave Oscillators*. Hewlett Packard, Palo Alto, CA, Sept. 1985.

Werner Anzill (S'92) was born in Freiburg, Germany, in 1965. He received the Dipl.Ing. degree at the Technische Universität in München, Germany in 1990.

Since then he has been working at the Lehrstuhl für Hochfrequenztechnik at the Technische Universität, München, in the field of noise in nonlinear circuits and is currently pursuing the Ph.D. degree.



Peter Russer (SM'81) was born in Vienna, Austria, in 1943. He received the Dipl.-Ing. degree in 1967 and the Dr. Tech. degree in 1971, both in electrical engineering and both from the Technische Universität, Vienna, Austria.

From 1968 to 1971 he was an Assistant Professor at the Technische Universität in Vienna. In 1971 he joined the Research Institute of AEG–Telefunken, Ulm, Germany, where he worked on fiber–optic communication, high-speed solid-state electronic circuits, laser modulation, and fiber–optic gyroscopes. In 1979 he was corecipient of the NTG award. Since 1981 he has held the chair of Hochfrequenztechnik at the Technische Universität München, Germany. Since 1992 he has been also the Director of the Ferdinand-Braun-Institut für Höchstfrequenztechnik, Berlin, Germany. His current research interests are microwave circuits, methods for computer-aided design of microwave circuits, electromagnetic fields, statistical noise analysis of microwave circuits, integrated microwave and millimeterwave circuits and microwave oscillators. He is the author of more than 150 papers in these areas.

Dr. Russer is a member of the German Informationstechnische Gesellschaft and the Austrian and German Physical Societies. In 1990 he was a Visiting Professor at the University of Ottawa, Canada.